NONLINEAR MODEL FOR A SHELL WITH DEFORMABLE TRANSVERSE FIBERS

L. I. Shkutin UDC 539.30

A nonlinear two-dimensional shell model of Timoshenko type [1, 2] is here generalized by the conservation of the transverse normal stress. The deformation in the generalized model is subject to a kinematic relation, which provides uniformity throughout the thickness in the transverse tension-compression deformation, and we use the method of [3, 4] for explicitly isolating the finite-rotation field.

As a result, the three-dimensional nonlinear problem for a shell is split up into two problems solved in sequence: a two-dimensional nonlinear problem defining the longitudinal components of the stress and strain tensors and a one-dimensional linear problem in terms of the transverse coordinate that defines the transverse components of those tensors.

A difference from [1, 2, 4] is that the differential order of the two-dimensional nonlinear problem is 12, while the number of natural contour conditions is six.

We use the symbols of [4]. The upper-case Latin subscripts take the values 1, 2, and 3, while the lower-case ones take the values 1 and 2.

1. Formulation of the Two-Dimensional Kinematic and Dynamic Equations. Let t_M be a three-dimensional curvilinear coordinate system related to the base surface b of the shell (the parameters t_1 and t_2 are the internal coordinates of this surface, while parameter t_3 is the coordinate normal to it). This coordinate system is put into correspondence with two initial bases: a three-dimensional one $\mathbf{A}_{(N)}(t_M)$ defined in the entire volume of the shell and a two-dimensional one $\mathbf{a}_{(N)}(t_m)$ defined on the basis surface.

Deformation of the shell transforms the initial bases into the corresponding instantaneous bases $\mathbf{A}_{\{N\}}(t_M)$ and $\mathbf{a}_{\{N\}}(t_m)$ (a possible dependence on time is supposed but not explicitly indicated). From the overall transformation, we isolate the rigid rotation that generates the rotated bases $\mathbf{A}_{[N]}(t_M)$ and $\mathbf{a}_{[N]}(t_m)$. The length of the normal vector $\mathbf{a}_{(3)}$ is taken as constant by definition (not necessarily unit vector). During the deformation, this vector is transformed to the instantaneous vector $\mathbf{a}_{\{3\}}$, which is not normal to the deformed basis surface and which has a length differing from the initial one. The corresponding turned vector $\mathbf{a}_{[3]}$ is taken as collinear with the instantaneous one by definition, so

$$\mathbf{a}_{\{3\}} = (a_{33} + u_{[33]})\mathbf{a}^{[3]}, \quad \{\mathbf{a}_{[3]} \| \mathbf{a}_{\{3\}} \Rightarrow \mathbf{a}^{[3]} \| \mathbf{a}_{\{3\}}\}.$$

The scalar function $u_{[33]}(t_m)$ is a measure of the elongation of the normal vector on deformation.

Let the deformation at any instant be subject to the kinematic relation

$$\mathbf{A}_{\{3\}} = \mathbf{a}_{\{3\}}, \tag{1.1}$$

which provides transverse tension-compression deformation homogeneous throughout the thickness of the shell. A consequence of (1.1) is that the field of displacements is linearly distributed over the normal coordinate:

$$\mathbf{U} = \mathbf{u} + t_3\mathbf{w}, \ \mathbf{w} = \mathbf{a}_{\{3\}} - \mathbf{a}_{(3)} \tag{1.2}$$

[$\mathbf{u}(t_m)$ is the displacement field for points in the basis plane]. From (1.4) of [4], we get the deformation field corresponding to the distribution of (1.2):

$$\mathbf{U}_{[n]} = \mathbf{u}_{[n]} + t_3\mathbf{w}_{[n]}, \ \mathbf{U}_{[3]} = \mathbf{u}_{[3]}. \tag{1.3}$$

Here $\mathbf{u}_{[N]}(t_m)$ and $\mathbf{w}_{[n]}(t_m)$ are the two-dimensional strain fields defined by

$$\mathbf{u}_{[n]} = \partial_n\mathbf{u} - (1/f)\mathbf{v}\times(\mathbf{a}_{(n)} + (1/2)\mathbf{v}\times\mathbf{a}_{(n)}),$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 168-174, January-February, 1984. Original article submitted October 6, 1982.

 0021-8944/84/2501-0154$08.50

$$\mathbf{w}_{[n]} = \mathbf{v}_{[n]}\times\mathbf{a}_{[3]} + \partial_n\mathbf{u}_{[3]},\ \mathbf{u}_{[3]} = \mathbf{a}_{\{3\}} - \mathbf{a}_{[3]},\ \mathbf{v}_{[n]} = (1/f)(\partial_n\mathbf{v} + (1/2)\mathbf{v}\times\partial_n\mathbf{v}),\quad f = 1 + (1/4)\mathbf{v}\cdot\mathbf{v} \tag{1.4}$$

($\mathbf{v}_{[n]}(t_m)$ is the field of bends in the coordinate lines t_m = const generated by the rigid-rotation field $\mathbf{v}(t_m)$, while ∂_n is the operator for partial differentiation with respect to t_n).

The variational principle representing the principle of virtual displacements for the shell with the kinematic relation of (1.1) employs two-dimensional dynamic equations defined on the basis surface b:

$$\nabla_{(n)}\mathbf{x}^{(n)} + \mathbf{f} = 0,\ \nabla_{(n)}\mathbf{z}^{(n)} - \mathbf{x}^{(3)} + \mathbf{h} = 0,\ \mathbf{a}_{\{N\}}\times\mathbf{x}^{(N)} + \partial_n\mathbf{a}_{\{3\}}\times\mathbf{z}^{(n)} = 0; \tag{1.5}$$

together with boundary conditions defined on the contour c

$$[(\mathbf{e}_{(\nu)}\cdot\mathbf{a}_{(n)})\mathbf{x}^{(n)} - \mathbf{f}_{(\nu)}]\cdot\nabla_0\mathbf{u} = 0,\ [(\mathbf{e}_{(\nu)}\cdot\mathbf{a}_{(n)})\mathbf{z}^{(n)} - \mathbf{h}_{(\nu)}]\cdot\nabla_0\mathbf{w} = 0; \tag{1.6}$$

a formula for the surface density of the virtual strain energy

$$w_0 = \mathbf{x}^{(N)}\cdot(\nabla_0\mathbf{u}_{[N]} - \mathbf{v}_0\times\mathbf{u}_{(N)}) + \mathbf{z}^{(n)}\cdot(\nabla_0\mathbf{w}_{[n]} - \mathbf{v}_0\times\mathbf{w}_{[n]}) \tag{1.7}$$

and equations defining the moments of the force fields in the shell:

$$\begin{gathered}\mathbf{x}^{(N)} = \frac{1}{j}\int_{b_-}^{b_+}\mathbf{X}^{(N)}J\,dt_3,\quad \mathbf{z}^{(n)} = \frac{1}{j}\int_{b_-}^{b_+}\mathbf{X}^{(n)}Jt_3\,dt_3,\\ \mathbf{f} = \frac{1}{j}\int_{b_-}^{b_+}\left[\mathbf{F}J + \partial_3(\mathbf{X}^{(3)}J)\right]dt_3,\quad \mathbf{f}_{(\nu)} = \frac{1}{j}\int_{b_-}^{b_+}\mathbf{F}_{(\nu)}J\,dt_3,\\ \mathbf{h} = \frac{1}{j}\int_{b_-}^{b_+}\left[\mathbf{F}Jt_3 + \partial_3(\mathbf{X}^{(3)}Jt_3)\right]dt_3,\quad \mathbf{h}_{(\nu)} = \frac{1}{j}\int_{b_-}^{b_+}\mathbf{F}_{(\nu)}Jt_3\,dt_3.\end{gathered} \tag{1.8}$$

Here $\nabla_{(n)}$ is the operator for covariant differentiation with respect to t_n in the initial basis $\mathbf{a}_{(N)}$, $\mathbf{e}_{(\nu)}$ is the field of unit normals to the end surface of the shell, $\mathbf{v}_0$ is the two-dimensional virtual-rotation field, ∇_0 is the total-variation operator, j and J are the Jacobians of the bases $\mathbf{a}_{(N)}$ and $\mathbf{A}_{(N)}$ correspondingly, $t_3 = b_-$ and $t_3 = b_+$ are the equations for the external surfaces of the shell, $\mathbf{X}^{(N)}(t_M)$ is the stress field in the shell, $\mathbf{F}(t_M)$ is the field bulk forces (including the inertial ones), and $\mathbf{F}_{(\nu)}(t_M)$ is the field of the external forces distributed over the end surface.

Equations (1.8) define the following: the two-dimensional field for the internal forces $\mathbf{x}^{(N)}$, the two-dimensional field for the internal moments $\mathbf{z}^{(n)}$, the two-dimensional fields for the external forces $\mathbf{f}$ and $\mathbf{f}_{(\nu)}$ (including the inertial ones), and the two-dimensional fields for the external moments $\mathbf{h}$ and $\mathbf{h}_{(\nu)}$ (including the inertial ones).

The second of the vector equations in (1.5) is converted to two equations by means of the third:

$$\begin{gathered}(\nabla_{(n)}\mathbf{z}^{(n)} - \mathbf{x}^{(3)} + \mathbf{h})\cdot\mathbf{a}^{[3]} = 0,\\ \nabla_{(n)}(\mathbf{a}_{\{3\}}\times\mathbf{z}^{(n)}) + \mathbf{a}_{\{n\}}\times\mathbf{x}^{(n)} + \mathbf{a}_{\{3\}}\times\mathbf{h} = 0,\end{gathered}$$

the first of which is the condition for equilibrium for the moments in the direction of the vector $\mathbf{a}_{\{3\}}$, while the second is the condition for equilibrium of the moments in directions normal to $\mathbf{a}_{\{3\}}$. The substitution

$$\mathbf{a}_{\{3\}}\times\mathbf{z}^{(n)} = \mathbf{y}^{(n)},\ \mathbf{a}_{\{3\}}\times\mathbf{h} = \mathbf{g}$$

converts this to the second equation of (2.6) in [4], which means that (1.5) contain the dynamic equations for a simpler model [4] as a particular case.

It is simplest to transfer to the scalar formulation of the two-dimensional kinematic and dynamic equations by means of the expansions

$$\begin{gathered}\mathbf{u} = u_{(N)}\mathbf{a}^{(N)},\ \mathbf{u}_{[n]} = u_{[nM]}\mathbf{a}^{[M]},\ \mathbf{u}_{[3]} = u_{[33]}\mathbf{a}^{[3]},\\ \mathbf{v} = v_{(N)}\mathbf{a}^{(N)},\ \mathbf{v}_{[n]} = v_{[nM]}\mathbf{a}^{[M]},\ \mathbf{w}_{[n]} = w_{[nM]}\mathbf{a}^{[M]},\end{gathered}$$

$$\mathbf{x}^{(n)} = x^{[nM]}\mathbf{a}_{[M]},\ \mathbf{f} = f^{[M]}\mathbf{a}_{[M]},\ \mathbf{f}_{(\nu)} = f_{(\nu M)}\mathbf{a}^{(M)},$$
$$\mathbf{z}^{(n)} = z^{[nM]}\mathbf{a}_{[M]},\ \mathbf{h} = h^{[M]}\mathbf{a}_{[M]},\ \mathbf{h}_{(\nu)} = h_{(\nu M)}\mathbf{a}^{(M)} \tag{1.9}$$

(the vector $\mathbf{u}_{[3]}$ by definition is a one-component one in the rotated basis). From (1.9) we formulate the following scalar kinematic and dynamic equations.

Equations defining the components of the strain tensors in terms of those of the displacement and rotation vectors:

$$u_{[nM]} = a^{LK}(\nabla_{(n)}u_{(L)} - w_{(nL)})(a_{MK} + w_{MK}),$$
$$v_{[n,M]} = \frac{1}{f}\nabla_{(n)}v_{(L)}\left(a^{LK} + \frac{1}{2}a^{LKJ}v_{(J)}\right)(a_{MK} + w_{(MK)}),$$
$$w_{[nm]} = (a_{33} + u_{[33]})\, a^{\cdot l3}_{m\cdot\cdot}v_{[nl]} + b_{nm}a^{33}u_{[33]}, \tag{1.10}$$
$$w_{[n3]} = \partial_n u_{[33]},\ f = 1 + \frac{1}{4}v_{(L)}v^{(L)},$$
$$w_{(NM)} = \frac{1}{f}a_{NML}v^{(L)} + \frac{1}{2f}(v_{(N)}v_{(M)} - a_{NM}v_{(L)}v^{(L)}).$$

Dynamic equations relating the components of the force tensors:

$$\nabla_{(n)}x^{[nM]} + a^{ML\cdot}_{\cdot\cdot K}v_{[nL]}x^{[nK]} + f^{[M]} = 0,$$
$$\nabla_{(n)}z^{[nM]} + a^{ML\cdot}_{\cdot\cdot K}v_{[nL]}z^{[nK]} - x^{[3M]} + h^{[M]} = 0, \tag{1.11}$$
$$a^{M\cdot\cdot}_{\cdot LK}\left[(a_{NM} + u_{[NM]})\, x^{[NL]} + (b_{nM} + w_{[nM]})\, z^{[nL]}\right] = 0.$$

The boundary conditions at the edge of the basis surface:

$$[e_{(\nu n)}x^{[nL]}(a_{LM} + w_{(LM)}) - f_{(\nu M)}]\nabla_0 u^{(M)} = 0,$$
$$[e_{(\nu M)}z^{[nL]}(a_{LM} + w_{(LM)}) - h_{(\nu M)}]\nabla_0 w^{(M)} = 0. \tag{1.12}$$

An equation defining the surface density of the virtual strain energy and which provides energy correspondence between the components of the strain and force tensors in the rotated basis:

$$w_0 = x^{[nM]}\nabla_0 u_{[nM]} + x^{[33]}\nabla_0 u_{[33]} + z^{[nM]}\nabla_0 w_{[nM]}. \tag{1.13}$$

In formulating problems on the moments of the stresses, we use the equations for the strain compatibility [4] instead of kinematic equations (1.10) for the strain tensors $u_{[nM]}$ and $v_{[nM]}$, as the former equations have the scalar representation

$$\nabla_{(n)}\hat{v}^{[nM]} + \frac{1}{2}a^{ML\cdot}_{\cdot\cdot K}v_{[nL]}\hat{v}^{[nK]} = 0,$$
$$\nabla_{(n)}\hat{u}^{[nM]} + a^{ML\cdot}_{\cdot\cdot K}(a_{nL} + u_{[nL]})\,\hat{v}^{[nK]} = 0, \tag{1.14}$$
$$\hat{v}^{[nM]} = a^{LM}a^{nm3}v_{[mL]},\ \hat{u}^{[nM]} = a^{LM}a^{nm3}u_{[mL]}.$$

We have used the following variation and differentiation rules for the vector and tensor fields in deriving (1.10)-(1.14):

$$\nabla_0\mathbf{a}_{[M]} = \mathbf{v}_0\times\mathbf{a}_{[M]},\ \nabla_0\mathbf{u}_{[N]} - \mathbf{v}_0\times\mathbf{u}_{[N]} = \mathbf{a}^{[M]}\nabla_0 u_{[NM]},$$
$$\nabla_{(n)}\mathbf{a}_{[M]} = \mathbf{v}_{[n]}\times\mathbf{a}_{[M]},\ \nabla_0\mathbf{w}_{[n]} - \mathbf{v}_0\times\mathbf{w}_{[n]} = \mathbf{a}^{[M]}\nabla_0 w_{[nM]},$$
$$\nabla_0 u_{[3n]} = 0,\ \nabla_{(n)}w_{(M)} = \partial_n w_{(M)} - c^{L}_{\cdot nM}w_{(L)},$$
$$\nabla_{(n)}y^{[mM]} = \partial_n y^{[mM]} + c^{m}_{\cdot nl}y^{[lM]} + c^{M}_{\cdot nL}y^{[mL]}$$

and have introduced the following symbols: a_{NM}, a_{NML} the metrical and discriminant tensors for the initial and rotated bases, b_{nm} a tensor for the initial curvature of the basis surface, $e_{(\nu n)} = \mathbf{e}_{(\nu)}\cdot\mathbf{a}_{(n)}$, $c^{K}_{\cdot nM}$ the Christofelians of the second kind for the initial basis defined by

$$c^{K}_{\cdot nM} = \begin{cases} \frac{1}{2} a^{kl} (\partial_n a_{ml} + \partial_m a_{ln} - \partial_l a_{nm}) & \{M = m,\ K = k\}, \\ -a^{33} b_{nm} & \{M = m,\ K = 3\}, \\ a^{kl} b_{nl} & \{M = 3,\ K = k\}, \\ 0 & \{M = 3,\ K = 3\}. \end{cases}$$

Kinematic equations (1.11) have been formulated apart from an arbitrary rotation with respect to the vector $\mathbf{a}_{\{3\}}$; to eliminate this arbitrary element, one must make the rotation vector subject to a scalar condition. The simplest is the condition $\mathbf{v}\cdot\mathbf{a}_{(3)}$, which rules out rigid rotation of the basis relative to the normal to the basal plane. For the same purpose one can use the conditions $u_{[12]} = u_{[21]}$ or $u_{[12]} = 0$ or $u_{[21]} = 0$, but to realize these in a nonlinear treatment is much more complicated than the condition

$$\mathbf{v}\cdot\mathbf{a}_{(3)} = \mathbf{v}_{(3)} = 0. \tag{1.15}$$

If necessary, the transition from the expansion in terms of the rotated basis to expansions in terms of the initial and instantaneous bases can be provided from the following basis-transformation formulas:

$$\mathbf{a}_{(N)} = (a_{MN} + w_{(MN)})\mathbf{a}^{[M]},\quad \mathbf{a}_{\{N\}} = (a_{NM} + u_{[NM]})\mathbf{a}^{[M]}.$$

2. Formulation of the Two-Dimensional Definitive Equations. We assume that we know the equations defining the symmetrical three-dimensional stress tensor

$$X^{(NM)} = \mathbf{X}^{(N)}\cdot\mathbf{A}^{\{M\}}$$

in terms of the symmetrical Green's three-dimensional strain tensor

$$U_{(NM)} = \frac{1}{2}(\mathbf{A}_{[M]}\cdot\mathbf{U}_{[N]} + \mathbf{A}_{[N]}\cdot\mathbf{U}_{[M]} + \mathbf{U}_{[N]}\cdot\mathbf{U}_{[M]}), \tag{2.1}$$

i.e., assume as known the three-dimensional definitive equations of the form

$$X^{(NM)} = X^{(NM)}(U_{(LK)}). \tag{2.2}$$

The three-dimensional tensor $U_{(LK)}$ is handled by means of (2.1), (1.3), (1.9), and (1.10) to be expressed via the two-dimensional parameters $u_{[lK]}$, $u_{[33]}$, $w_{[lK]}$, so the bulk density of virtual strain energy defined by

$$W_0 = X^{(NM)}\,\nabla_0 U_{(NM)}, \tag{2.3}$$

can be represented as

$$W_0 = X^{(NM)}[(\partial U_{(NM)}/\partial u_{[lK]})\nabla_0 u_{[lK]} + (\partial U_{(NM)}/\partial u_{[33]})\nabla_0 u_{[33]} + (\partial U_{(NM)}/\partial w_{[lK]})\nabla_0 w_{[lK]}].$$

The surface density of the virtual strain energy is determined via the bulk density from

$$w_0 = \frac{1}{j}\int_{b_-}^{b_+} w_0 J\, dt_3.$$

Comparison of this formula with (1.13) gives the two-dimensional definitive equations:

$$\begin{aligned} x^{[lK]} &= \frac{1}{j}\int_{b_-}^{b_+} (\partial U_{(NM)}/\partial u_{[lK]})\, X^{(NM)} J\, dt_3, \\ x^{[33]} &= \frac{1}{j}\int_{b_-}^{b_+} (\partial U_{(NM)}/\partial u_{[33]})\, X^{(NM)} J\, dt_3, \\ z^{[lK]} &= \frac{1}{j}\int_{b_-}^{b_+} (\partial U_{(NM)}/\partial w_{[lK]})\, X^{(NM)} J\, dt_3. \end{aligned} \tag{2.4}$$

These equations close the system formulated in Sec. 1 of two-dimensional kinematic and dynamic equations (1.10)-(1.12) and (1.15). The closed system is formed by the thirty-four equations for the functions $u_{(N)}$, $v_{(N)}$, $u_{[33]}$, $u_{[nM]}$, $w_{[nM]}$, $x^{[NM]}$, $z^{[nM]}$. Sixteen of these equations are algebraic. As a result of eliminating them, we get a system of 18 first-order differential equations for the functions $u_{(N)}$, $v_{(n)}$, $u_{[33]}$, $x^{[nM]}$, $z^{[nM]}$. This system allows us to eliminate the functions $x^{[nM]}$, $z^{[nM]}$ and leads to a system of six second-order equations for the kinematically unknown parameters $u_{(N)}$, $v_{(N)}$, $u_{[33]}$. The complete differential order of the decision system in each of the variables t_m is 12, while the number of contour boundary conditions in (1.12) is six.

3. Definition of the Three-Dimensional Stress and Strain Tensors. On solving the two-dimensional problem formulated by the closed system (1.10)-(1.12), (1.15), and (2.4), we determine the two-dimensional kinematic parameters: the displacement vector $u_{(N)}$ and the rotation vector $v_{(N)}$ ($v_{(3)} = 0$), the first strain tensor $u_{[NM]}$ ($u_{[3m]} = 0$), and the second strain tensor $w_{[nM]}$. From (1.2), (1.3), and (2.1) one then derives the three-dimensional kinematic parameters: the displacement vector $\mathbf{U}(t_M)$ and the symmetrical strain tensor $U_{(NM)} \times (t_L)$. Equations (2.2) define the symmetrical stress tensor $X^{(NM)}(t_L)$. However, the determination of the components $X^{(N3)} = X^{(3N)}$ in this way cannot be considered as satisfactory, since it does not provide obedience to the conditions at the surfaces bounding the shell $t_3 = b_-(t_m)$ and $t_3 = b_+(t_m)$. More accurate determination of these components is provided by the following method, which is based on asymptotic analysis of the linear elasticity problem for a shell [5]: we determine the components $X^{(NM)}$ from the coupling equations together with the corresponding stress vectors $\mathbf{X}^{(n)} = X^{(nM)}A_{(M)}$, while the vector $\mathbf{X}^{(3)} = X^{(3M)}A_{(M)}$ is determined from dynamic equation (1.5) of [4] by quadrature with respect to the normal coordinate t_3:

$$\mathbf{X}^{(3)} = \frac{1}{J}\left\{\mathbf{f}^{(3)} - \int\left[\mathbf{F}J + \partial_n(\mathbf{X}^{(n)}J)\right]dt_3\right\}, \tag{3.1}$$

where the vector $\mathbf{f}^{(3)}(t_m)$ is determined by subordinating the vector $\mathbf{X}^{(3)}(t_M)$ to a boundary condition at one of the outer surfaces [obedience to it at the other surface is provided by the equations (1.5)].

This gives the complete three-dimensional stress tensor. By inverting (2.2) one can derive the corresponding three-dimensional strain tensor. This completes the procedure for solving the nonlinear three-dimensional deformation problem.

The nonlinear model represented by (1.10)-(1.12), (1.15), (2.4), and (3.1) is free from the three major disadvantages in the model of [1, 2].

Firstly, it enables one by solving the two-dimensional problem to obey the kinematic boundary condition $\mathbf{U} = 0$ in the three-dimensional problem exactly at the end surface, whereas a contour condition was lacking for this in the two-dimensional treatment of [1, 2].

Secondly, equations (2.4) in the two-dimensional treatment are unambiguously formulated in terms of the definitive equations (2.2) for the three-dimensional one and therefore differ from [1, 2] in not containing undefined shear coefficients.

Thirdly, the transverse components of the stress tensor are determined from (3.1) as single-valued functions of the normal coordinate that satisfy the boundary conditions at the outer surfaces, whereas in [1, 2] these components contain undefined functions of the normal coordinate as factors.

4. Transfer to Physical Components. Let t_N be the principal coordinate system of the shell, while u_N and v_N are the physical components of the displacement and rotation vectors in the initial basis, and u_{NM}, v_{nM}, w_{nM}, x_{NM}, z_{nM} are the physical components of the strain and force tensors in the rotated basis.

The following groups of two-dimensional equations form a closed system for the physical components of the vectors and tensors defining the nonlinear strain in the shell with the kinematic relation of (1.1).

Equations defining the components of the first strain tensor in terms of the displacements and rotations:

$$\begin{gathered}
u_{nL} \mp (u_{(nM)} - w_{(nM)})(e_{LM} + w_{(LM)}), \\
u_{(nM)} = [\partial_n(a_M u_M) - c_{LnM}a_L u_L]/a_n a_M, \\
w_{(NM)} = \frac{1}{f}e_{NML}v_L + \frac{1}{2f}(v_N v_M - e_{NM}v_L v_L),
\end{gathered}$$

$$f = 1 + \frac{1}{4} v_L v_L, \; v_3 = 0. \tag{4.1}$$

Equations defining the components of the bending tensor in terms of the rotations:

$$v_{nJ} = \frac{1}{f} v_{(nM)} \left(e_{ML} + \frac{1}{2} e_{MLK} v_K \right) (e_{JL} + w_{(JL)}),$$
$$v_{(nM)} = [\partial_n (a_M v_M) - c_{LnM} a_L v_L)/a_n a_M, \; v_3 = 0. \tag{4.2}$$

Equations defining the components of the second strain tensor in terms of the components of the bending tensor and the relative extension of the normal u_{33}:

$$w_{nM} = e_{3Ml} v_{nl} (1 + u_{33}) + [\partial_n (a_M u_{3M}) - c_{LnM} a_L u_{3L}]/a_n a_M. \tag{4.3}$$

The equations of motion (equilibrium):

$$\nabla_n x_{nM} + e_{MLK} v_{nL} x_{nK} + f_M = 0,$$
$$\nabla_n z_{nM} + e_{MLK} v_{nL} z_{nK} - x_{3M} + h_M = 0, \tag{4.4}$$
$$e_{MLk} [(e_{NM} + u_{NM}) x_{NL} + (b_{nM} + w_{nM}) z_{nL}] = 0.$$

The equation for the surface density of the virtual strain energy:

$$w_0 = x_{nM} \nabla_0 u_{nM} + x_{33} \nabla_0 u_{33} + z_{nM} \nabla_0 w_{nM}. \tag{4.5}$$

The boundary conditions at the contour of the basis surface can have either a dynamic formulation

$$e_{(\nu n)} x_{nL} (e_{LM} + w_{(LM)}) = f_{(\nu M)}, \; e_{(\nu n)} z_{nL} (e_{LM} + w_{(LM)}) = h_{(\nu M)}, \tag{4.6}$$

when the stresses are given at the end surface, or a kinematic one

$$\nabla_0 u_M = 0, \; \nabla_0 w_M \equiv \nabla_0 (u_{3M} - w_{(M3)}) = 0, \tag{4.7}$$

when the displacements are given at this surface, or else a mixed formulation, when the stresses are given on one part of that surface and the displacements on the other.

Summation on the coupled subscripts is involved in (4.1)-(4.7); a_n are the metrical Lamé parameters for the undeformed basis surface, b_{nn} are the principal curvatures of this, $e_{(\nu n)}$ are the direction cosines of the normal to the edge, $f_{(\nu M)}$ are the physical components of the principal vector and principal moment for the end load in the initial basis, f_M and h_M are the physical components of the principal vector and principal moment for the surface load in the rotated basis, e_{NM} is the Kronecker tensor, e_{NML} is the Levi-Civita tensor, and

$$a_3 = 1, \; b_{12} = b_{21} = b_{n3} = 0, \; u_{3n} = 0, \; c_{nnn} = \partial_n a_n / a_n, \; c_{112} = c_{121} = \partial_2 a_1 / a_1,$$
$$c_{mn3} = a_n b_{nm} / a_m, \; c_{212} = c_{221} = \partial_1 a_2 / a_2, \; c_{mnn} = -a_n \partial_m a_n / (a_m)^2 (m \neq n),$$
$$c_{3nm} = -a_n a_m b_{nm}, \; c_{3n3} = 0,$$
$$\nabla_n y_{nM} = \frac{a_M}{a_1 a_2} \left[\partial_n \left(\frac{a_1 a_2}{a_n a_M} y_{nM} \right) + c_{MnL} \frac{a_1 a_2}{a_n a_L} y_{nL} \right].$$

The physical components of the three-dimensional stress and strain tensors are defined in the metric of the initial basis by

$$X_{NM} = A_N A_M X^{(NM)}, \; U_{NM} = U_{(NM)} / A_N A_M,$$

where

$$A_3 = 1; \; A_n = a_n B_n; \; B_n = 1 + b_{nn} t_3.$$

Then the physical tensors X_{NM} and U_{NM} remain symmetrical and correspond in energy to one another in the sense of (2.3).

The three-dimensional strain tensor is determined from the two-dimensional ones by

$$U_{nm} = \frac{1}{2} \left[B_n^{-1} (u_{nm} + t_3 w_{nm}) + B_m^{-1} (u_{mn} + t_3 w_{mn}) + B_n^{-1} B_m^{-1} (u_{nL} + t_3 w_{nL}) (u_{mL} + t_3 w_{mL}) \right],$$
$$U_{n3} = U_{3n} = \frac{1}{2} B_n^{-1} (1 + u_{33}) (u_{n3} + t_3 w_{n3}), \quad U_{33} = u_{33} + \frac{1}{2} (u_{33})^2. \tag{4.8}$$

With a known functional relationship between the X_{NM} and U_{LK}, one has the following linkage equations between the two-dimensional force and strain tensors:

$$x_{lK} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial u_{lK}) X_{NM} B_1 B_2 dt_3,$$

$$x_{33} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial u_{33}) X_{NM} B_1 B_2 dt_3, \tag{4.9}$$

$$z_{lK} = \int_{b_-}^{b_+} (\partial U_{NM}/\partial w_{lK}) X_{NM} B_1 B_2 dt_3.$$

Equations (4.1)-(4.9) form a closed system for the unknown functions u_N, v_n, u_{33}, u_{nM}, w_{nM}, x_{NM}, z_{nM} and their first-order partial derivatives. The bending tensor v_{nM} plays an auxiliary role in this system of abbreviated denotation for differential expression (4.2).

When the two-dimensional system of (4.1)-(4.9) has been solved, the three-dimensional parameters of the state of stress and strain in the shell are determined from the scheme presented in Sec. 3.

LITERATURE CITED

1. L. Ya. Ainola, "A nonlinear theory of Timoshenko type for elastic shells," Izv. Akad. Nauk ESSR, Ser. Fiz.-Mat. Tekh. Nauk, 14, No. 3 (1965).
2. K. Z. Galimov, "A nonlinear theory of thin shells of Timoshenko type," in: Researches on the Theory of Plates and Shells [in Russian], Issue 11, Kazan' (1975).
3. L. I. Shkutin, "Nonlinear models for deformable media containing moments," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1980).
4. L. I. Shkutin, "A nonlinear model for a shell with undeformable transverse fibers," Zh. Prikl. Mekh. Tekh. Fiz., No. 1 (1982).
5. N. A. Fedorova and L. I. Shkutin, "The asymptotic behavior in an axially symmetrical elasticity problem for an anisotropic cylindrical shell," Zh. Prikl. Mekh. Tekh. Fiz., No. 5 (1981).

EXISTENCE OF SOLUTIONS IN IDEAL HENCKE PLASTICITY

A. M. Khludnev

UDC 539.214+539.374+517.9

The existence of a weak solution in the theory of ideal Hencke plasticity is obtained only in the particular case of the Mises flow condition and under the assumption of isotropy of the material [1]. The strain vector is here found from a space conjugate to $L^\infty(\Omega)$. The existence of a solution for an arbitrary flow condition and without the assumption of isotropy is proved in this paper. The displacement vector belongs to the space $L^{3/2}(\Omega)$.

The governing equations of the plasticity theory under consideration yield a representation of the total strains in the form of a sum of elastic and plastic components

$$\varepsilon_{ij}(u) = c_{ijkl}\sigma_{kl} + \xi_{ij}, \quad i, j = 1, 2, 3, \tag{1}$$

where the stresses do not exceed the yield point $\Phi(\sigma) \leqslant 0$, while the plastic strains ξ_{ij} satisfy the inequality [1-3]

$$\xi_{ij}(\tau_{ij} - \sigma_{ij}) \leqslant 0 \ \forall \tau, \ \Phi(\tau) \leqslant 0. \tag{2}$$

The equilibrium equations are satisfied in the domain $\Omega \subset R^3$

$$-\sigma_{ij,j} = f_i, \quad i = 1, 2, 3. \tag{3}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 174-177, January-February, 1984. Original article submitted November 1, 1982.

0021-8944/84/2501-0160$08.50